TWO SPACE-CONSTRAINED MULTI-ITEM DETERMINISTIC INVENTORY MODELS.

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THESIS

TWO SPACE-CONSTRAINED MULTI-ITEM DETERMINISTIC INVENTORY MODELS

by

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Two Space-Constrained Multi-Item Deterministic Inventory Models

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I. INTRODUCTION

Military forces have to be supplied with all kinds of items in order to complete their missions. Because many of the needed items cannot be purchased from outside suppliers at the time the forces need them, the items have to be purchased and stored earlier than actually needed.

From the point of view of the ultimate user it would be best to store all such needed goods as near as possible to his own location, so that whenever an item is needed it can be delivered immediately. However, there are obviously constraints which make this course of action impossible.

First, there is a budget constraint which allows only a certain amount of money to be spent on purchases of equipment, spare parts and other needed goods and, second, there is a space constraint because the supply support units are required to be mobile so that they can stay in close proximity to the combat units they support. In the case of direct support units for land forces the budget constraint is either non-existent or overwhelmed by the space constraint.

In the Federal German Army these units are called BVP's and each consits of a fleet of trucks. They are equivalent to the Direct Support Units (DSU's) of the U. S. Army. The BVP's constitute the last echelon of the German Army's supply system and are not involved in buying goods from outside suppliers. Therefore the budget constraint on purchases is

nonexistent. However, the number of available trucks for a BVP is limited by its operational requirements and, to a certain extent, by the available budget for the armed services. Therefore any relevant inventory model has to take a space constraint into consideration when calculating optimal operating policies.

Two models will be analyzed in this thesis. The first will be a multi-item deterministic lot size model subject to a space restriction which requires that the amount in inventory at any time cannot exceed the available storage space and a restriction that the time between orders for all items must be the same. The variables which are being optimized in this system are the time between orders for each item and the times between orders for different items. The notion of allowing orders for different items to arrive at different times will be referred to as time phasing of the orders for different items. The solution for this model is obtained for the two-item case and the procedure for solving the n-item case is outlined. The second model is a multi-item deterministic order level system. It is subject to the same space and cycle time constraints as the lot size model. However, stockouts are allowed. The decision variables are the order levels of each of the items and the time phasing of the ordering of each item. This model is solved for the two-item case and discussed for the n-item case.

II. PREVIOUS WORK IN THE AREA

A. CONSTRAINED LOT SIZE MODEL

A multi-item lot size model with a single space constraint has been examined by Churchman [2] and Holt [6]. The model was defined in the usual form of deterministic inventory problems; that is, to minimize the total variable costs – reorder plus carrying costs – of the system. The space constraint required that the sum of the average space requirements for all items not exceed the total space available for these items. The assumptions which were made in the formulation of the problem were:

1. a known constant demand rate;
2. infinite replenishing rate;
3. constant lead time;
4. shortages are not allowed.

The cycle lengths of all items were not required to be identical. In addition, the time phasing of each item was not addressed.

A disadvantage of this model is that only the sum of the average inventories must be no greater than the space available. This requirement can be satisfied mathematically and yet not be met in reality because the possible different cycle lengths may cause the order for more than one item to arrive at the same time.

B. A CONSTRAINED ORDER LEVEL SYSTEM

A multi-item deterministic order level model has been examined by Naddor [7]. This model used the first three assumptions of the above model by Holt and, in addition, required a fixed review interval for all items. Backorders were allowed however. The objective function for minimization was the total variable cost — carrying plus backorder costs — and the constraint required that the summation of the space requirements of the optimum order levels not exceed the total space available.

The major difference between Naddor's model and the one to be presented in this paper is that in his model all orders arrive at the same time, while the model being developed here permits time phasing of arrivals of orders of different items. His solution for the optimal order level for the n-item case is:

$$S_i = q_i \frac{c_{2i} - g_o b_i}{c_{1i} + c_{2i}}$$

where

S_i = order level of item i when lead time is zero;

q_i = replenishment size of item i;

c_{1i} = the carrying cost of item i per unit of time;

c_{2i} = the shortage cost of item i per unit of time;

b_i = unit space required for item i;

B = total warehouse space available;

and $$g_o = \left(\sum_{i=1}^{n} \frac{c_{2i} b_i q_i}{c_{1i} + c_{2i}} - B \right) \Big/ \sum_{i=1}^{n} \frac{b_i^2 q_i}{c_{1i} + c_{2i}} \quad .$$

III. A CONSTRAINED MULTI-ITEM LOT SIZE SYSTEM

The model which is developed in this chapter is an n item lot size model constrained by the amount of space which is available for storage of inventory. The following assumptions are made:

1. Demand for item i is deterministic at a constant positive rate of r_i units per unit time.

2. No shortages are allowed and the stock of an item will be immediately replenished whenever its inventory level reaches zero.

3. The lead time is constant.

4. The replenishment rate is infinite.

5. One type of storage is usable by all items.

6. The inventory carrying cost is proportional to the average amount in inventory at any time; its dimension is \$ per unit-time.

7. The order cost is a constant which is incurred each time an order is placed; its dimension is \$.

The variables which will be optimized to produce minimum cost are the length T of the reorder period for every item and the time x_i at which an order for the ith item arrives after arrival of an order of item number 1 ($x_1 = 0$). The cost equation, as a function of the scheduling period for any single item under these assumptions, is given by

$$C_i(T) = c_{1i} r_i T/2 + c_{3i}/T \; ; \qquad (1)$$

where c_{1i} = the inventory carrying cost, and

c_{3i} = the unit reorder cost.

This notation is that of Ref. 7, page 58. For n items, the total cost can be expressed as

$$TC = \sum_{i=1}^{n} (c_{1i} r_i T/2 + c_{3i}/T) \quad ,$$

given that all items are using the same reorder period length.

The space constraint can be formulated as

$$\sum_{i=1}^{n} b_i I_i(t) \leq B \qquad \text{for all } t \geq 0 \ , \tag{2}$$

where

B = total space available;

b_i = the space occupied by a unit of the ith item;

$I_i(t)$ = the inventory level of the ith item at time t.

Since maximum inventory levels occur at the times at which orders are received, these times correspond to critical points relative to the constraint. Therefore the constraint needs only to be examined at these arrival times. This will result in n constraints for an n-item case because of the possibility that each item may arrive at a different time.

To solve the problem let λ_i represent the ith Lagrange multiplier. Then the Lagrangian becomes

$$L = \sum_{i=1}^{n} C_i(T) + \sum_{i=1}^{n} \lambda_i F_i \tag{3}$$

where $C_i(T)$ is given by equation (1) and F_i represents the ith constraint function (describing (2) at each x_i, $i=1,2,\ldots,n$). The solution to equation (3) will be presented for cases which have two and n items. The two-item case is presented to illustrate the general solution procedure.

A. SOLUTION OF THE TWO-ITEM CASE

Equation (3) for the two-item case reduces to:

$$L = c_{11}r_1T/2 + c_{31}/T + c_{12}r_2T/2 + c_{32}/T + \lambda_1F_1 + \lambda_2F_2 \ .$$

Typical fluctuations of the inventory levels of the items can be represented as shown in Figure 1.

From Figure 2 it can be seen that the constraint function at time zero when item number 1 arrives will be

$$b_1r_1T + b_2r_2x_2 \leq B \quad ,$$

and that the constraint function at time x_2 when item number 2 arrives will be

$$b_1r_1(T - x_2) + b_2r_2T \leq B \quad .$$

Conversion of inequalities into equalities for use in the Lagrangian yields:

$$F_1 = b_1r_1T + b_2r_2x_2 + V_1^2 - B \quad ; \tag{4}$$

$$F_2 = b_1r_1(T - x_2) + b_2r_2T + V_2^2 - B \quad ; \tag{5}$$

where V_1 and V_2 are slack variables.

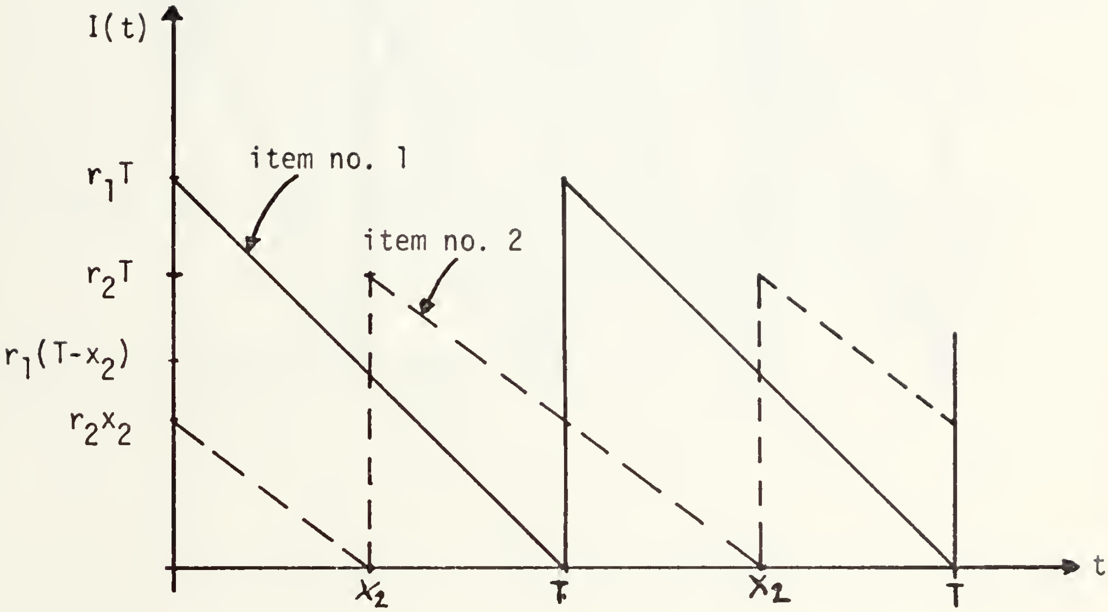


FIGURE 1: Fluctuation of inventory levels for two items for the deterministic lot size model.

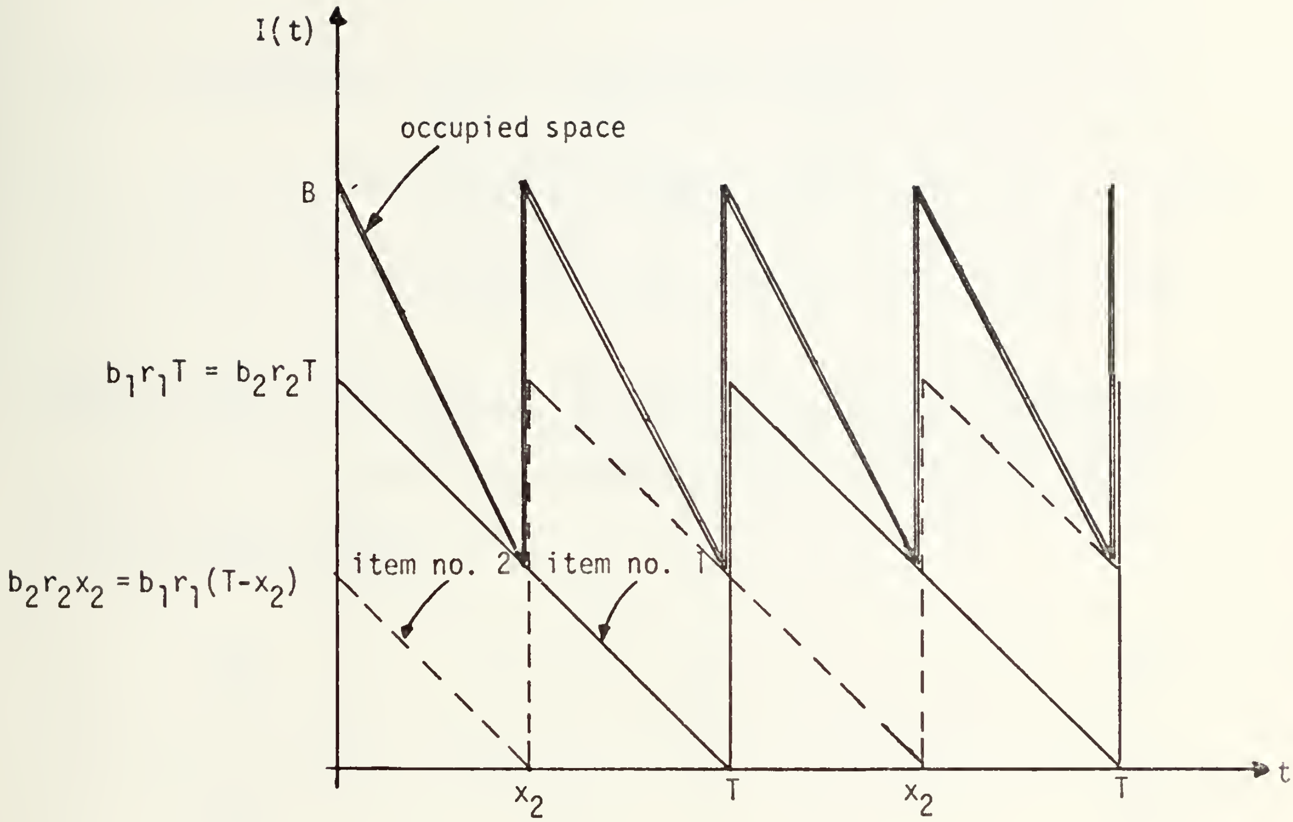


FIGURE 2: Space occupied by two items when $b_1r_1 = b_2r_2$

Combining (3), (4) and (5) gives:

$$\begin{aligned} L = {} & c_{11}r_1T/2 + c_{31}/T + c_{12}r_2T/2 + c_{32}/T \\ & + \lambda_1(b_1r_1T + b_2r_2x_2 + V_1^2 - B) \\ & + \lambda_2(b_1r_1(T-x_2) + b_2r_2T + V_2^2 - B) \ . \qquad (6) \end{aligned}$$

Taking partial derivatives of the Lagrangian with respect to V_1, V_2, x_2, T, λ_1 and λ_2, setting them equal to zero, and simplifying, yields:

$$\lambda_1V_1 = 0 \quad ; \qquad (7)$$

$$\lambda_2V_2 = 0 \quad ; \qquad (8)$$

$$\lambda_1b_2r_2 - \lambda_2b_1r_1 = 0 \quad ; \qquad (9)$$

$$\begin{aligned} & c_{11}r_1/2 - c_{31}/T^2 + c_{12}r_2/2 - c_{32}/T^2 + \lambda_1b_1r_1 \\ & \qquad + \lambda_2b_1r_1 + \lambda_2b_2r_2 = 0 \quad ; \qquad (10) \end{aligned}$$

$$b_1r_1T + b_2r_2x_2 + V_1^2 - B = 0 \quad ; \qquad (11)$$

$$b_1r_1(T-x_2) + b_2r_2T + V_2^2 - B = 0 \quad . \qquad (12)$$

Equations (7), (8) and (9) indicate that there are only two possible cases:

1. $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ which implies that $V_1 = V_2 = 0$;

2. $\lambda_1 = \lambda_2 = 0$, and the V_i values can be arbitrary.

When $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, both constraints are binding. Solving equations (11) and (12) with $V_1 = V_2 = 0$ for T and x_2 respectively, yields:

$$T^* = B(b_1r_1 + b_2r_2)/(b_1^2r_1^2 + b_1r_1b_2r_2 + b_2^2r_2^2) \quad . \tag{13}$$

$$x_2^* = Bb_2r_2/(b_1^2r_1^2 + b_1r_1b_2r_2 + b_2^2r_2^2) \quad . \tag{14}$$

Substituting T and x_2 into the total cost equation yields the minimum cost C_o where

$$C_o = B(c_{11}r_1 + c_{12}r_2)(b_1r_1 + b_2r_2)/2D + D(c_{31} + c_{32})/A \ ,$$

and

$$D = b_1^2r_1^2 + b_1r_1b_2r_2 + b_2^2r_2^2 \quad ;$$

$$A = B(b_1r_1 + b_2r_2) \quad .$$

The ratio

$$\frac{x_2^*}{T^*} = \frac{b_2r_2}{b_1r_1 + b_2r_2}$$

provides a convenient means of comparing x_2 to T for this completely constrained case. If $b_1r_1 = b_2r_2$ then $\frac{x_2^*}{T^*} = \frac{1}{2}$. If $b_1r_1 < b_2r_2$ then $\frac{x_2^*}{T^*} > \frac{1}{2}$. If $b_1r_1 > b_2r_2$ then $\frac{x_2^*}{T^*} < \frac{1}{2}$.

For the unconstrained case, equation (9) requires that if one of the λ values equals zero, the other has to be zero too. The optimal unconstrained value of T, denoted as T_u, can be determined from equation (10). It is

$$T_u = \sqrt{\frac{2(c_{31} + c_{32})}{c_{11}r_1 + c_{12}r_2}} \quad . \tag{15}$$

Equation (15) is the same expression as derived by Naddor [7]. The value for x_2 is dependent, however, on the value of B. The value of B which corresponds to the dividing point between cases 1 and 2 is obtained by equating equations (13) and (15) and solving for B. This value is denoted as B_1, and is given by equation (16)

$$B_1 = \frac{D}{b_1r_1 + b_2r_2} T_u \quad . \tag{16}$$

When $B \leq B_1$, equations (13) and (14) give the values of T^* and x_2^*. When $B > B_1$, $T^* = T_u$ and the value of x_2 is no longer restricted to that given by (14).

If $B \geq B_2$ where $B_2 = (b_1 r_1 + b_2 r_2)\ T_u$, then x_2^* can be any value in the interval $0 \leq x_2 \leq T_u$. This corresponds to B being so large that even if orders for both items arrive simultaneously they will not be faced with a storage restriction. When $B_1 < B < B_2$, the storage constraint will not be restrictive provided that x_2 is so selected that the constraint is not exceeded at the time either item arrives.

When the first item arrives (at t = 0) the total combined on-hand inventory consumes $b_1 r_1 T_u + b_2 r_2 x_2$ amount of storage space. When the second item arrives (at $t = x_2$) the total combined on-hand inventory consumes space in the amount of $b_1 r_1 (T_u - x_2) + b_2 r_2 T_u$. The value of x_2 must be selected so that the total combined on-hand inventory at either time does not exceed B. Thus if x_2 lies in the interval

$$\frac{B_2 - B}{r_1 b_1} \leq x_2 \leq T_u - \frac{B_2 - B}{r_2 b_2} \quad , \tag{17}$$

then both conditions will be satisfied when $B_1 < B < B_2$.

When $B = B_2$ the interval given by (17) simplifies to $0 \leq x_2 \leq T_u$. As B decreases below B_2, the upper bound decreases and the lower bound increases at the same rate. The interval reduces to a single value for x_2 at $B = B_1$; the corresponding x_2 value is given by equation (14). As a consequence of this behavior of the interval for B between B_1 and B_2, it follows that equation (14) gives an optimal value of x_2 for all $B > B_1$, and the only optimal value for $B \leq B_1$.

B. SOLUTION OF THE n-ITEM CASE

The Lagrangian for n items is:

$$L = \sum_{i=1}^{n} (c_{1i} r_i T/2 + c_{3i}/T) + \sum_{i=1}^{n} \lambda_i F_i \ , \tag{18}$$

with the constraint functions given below. These functions assume $0 \leq x_2 \leq x_3 \leq \cdots \leq x_n$.

$$\begin{array}{l}
b_1 r_1 T + b_2 r_2 x_2 + b_3 r_3 x_3 + \quad \cdots \quad + b_n r_n x_n + V_1^2 = B \\
b_1 r_1 (T - x_2) + b_2 r_2 T + b_3 r_3 (x_3 - x_2) + \quad \cdots \quad + b_n r_n (x_n - x_2) + V_2^2 = B \\
b_1 r_1 (T - x_3) + b_2 r_2 (T - (x_3 - x_2)) + b_3 r_3 T + \cdots + b_n r_n (x_n - x_3) + V_3^2 = B \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
b_1 r_1 (T - x_{n-1}) + b_2 r_2 (T - (x_{n-1} - x_2)) + \cdots + b_n r_n (x_n - x_{n-1}) + V_{n-1}^2 = B \\
b_1 r_1 (T - x_n) + b_2 r_2 (T - (x_n - x_2) + \quad \cdots \quad + b_n r_n T + V_n^2 = B
\end{array}$$

After forming the Lagrangian, taking partial derivatives with respect to $x_2, x_3, \ldots, x_n$ and $V_1, V_2, \ldots, V_n$, and setting them equal to zero, it can be shown that either all of the Lagrange multipliers are zero or all slack variables are zero. The partial derivatives with respect to $\lambda_1, \lambda_2, \ldots, \lambda_n$,

when the V_i's are all zero give back equality constraints at $t = 0, x_2, \ldots, x_n$. Solving these equations for T, x_2, ..., x_n, yields:

$$T^* = \frac{B \sum_{i=1}^{n} b_i r_i}{D} ; \tag{19}$$

$$x_i^* = \frac{B \sum_{k=2}^{i} b_k r_k}{D} , \quad i = 1,2,\ldots,n ; \tag{20}$$

where

$$D = \sum_{i=1}^{n} b_i^2 r_i^2 + \sum_{i=1}^{n-1} b_i r_i \sum_{j=i+1}^{n} b_j r_j . \tag{21}$$

The resulting value of C_o, the minimum total variable cost, is:

$$C_0 = B \left(\sum_{i=1}^{n} c_{1i} r_i/2 \; (b_1 r_1 + \ldots + b_n r_n \right)/D + D \left(\sum_{i=1}^{n} c_{31}/B \; (b_1 r_1 + \ldots + b_n r_n \right) .$$

In the unconstrained case, optimal T is obtained from

$$T_u = \sqrt{\frac{2 \sum_{i=1}^{n} c_{3i}}{\sum_{i=1}^{n} c_{1i} r_i}} . \tag{22}$$

The n-item equivalents of B_1 and B_2 are:

$$B_1 = \frac{D \; T_u}{\sum_{i=1}^{n} b_i r_i} ; \tag{23}$$

$$B_2 = \sum_{i=1}^{n} b_i r_i T_u \quad . \tag{24}$$

And, if $B \geq B_2$, then $T^* = T_u$ and each x_i can take on any value in the interval $0 \leq x_i \leq T_u$. When $B_1 < B < B_2$ then the x_i^*'s must be chosen in a way that the constraints are not violated. As B decreases below B_2, the upper bound decreases and the lower bound increases. However, equation (20) has been obtained for the case where the constraints have to be met, i.e. where we had equalities in all of the constraint functions. Changing the equalities to inequalities will result in an infinite set of optimal feasible solutions. But x_i as given by equation (20) will always be in this set of optimal feasible solutions because the solution to the more restricted problem will always be an answer to the less restricted problem.

The set of optimal solutions creates the ranges for the x_i values. They could be obtained using linear programming in the following way:

1. Possible objective functions are
   a) min x_i, $i = 2,3,\ldots,n$, to get the smallest lower bound for x_i;
   b) max x_i, $i = 2,3,\ldots,n$, to get the greatest upper bound for x_i.
2. The constraints are all of those shown below equation (18).

For solving problems which have $n > 2$ items the flow chart of Figure 3 is helpful and summarizes the preceding analysis.

C. EXAMPLES

Two examples are presented below to illustrate the model.

Example 1

Consider a warehouse which stores two different items whose parameters are:

Item 1:	Item 2:
$r_1 = 200$ units per year	$r_2 = 250$ units per year
$b_1 = 5\ ft^3$/unit	$b_2 = 3\ ft^3$/unit
$c_{11} = \$\ 2$ per unit-year	$c_{12} = \$\ 1$ per unit-year
$c_{31} = \$\ 30$	$c_{32} = \$\ 25$

If total available warehouse space is 400 cubic feet, what are the optimal order period T^* for each item and the optimal arrival time x_2^* for item number 2 after an order of item number 1 arrives?

Calculating T_u and B_1 yields:

$$T_u = 0.41 \text{ years} \quad ; \quad B_1 = 543\ ft^3 \quad .$$

Because $B < B_1$, both storage constraints are binding. Therefore we get $T^* = 0.30$ years and $x_2^* = 0.13$ years by using equations (13) and (14).

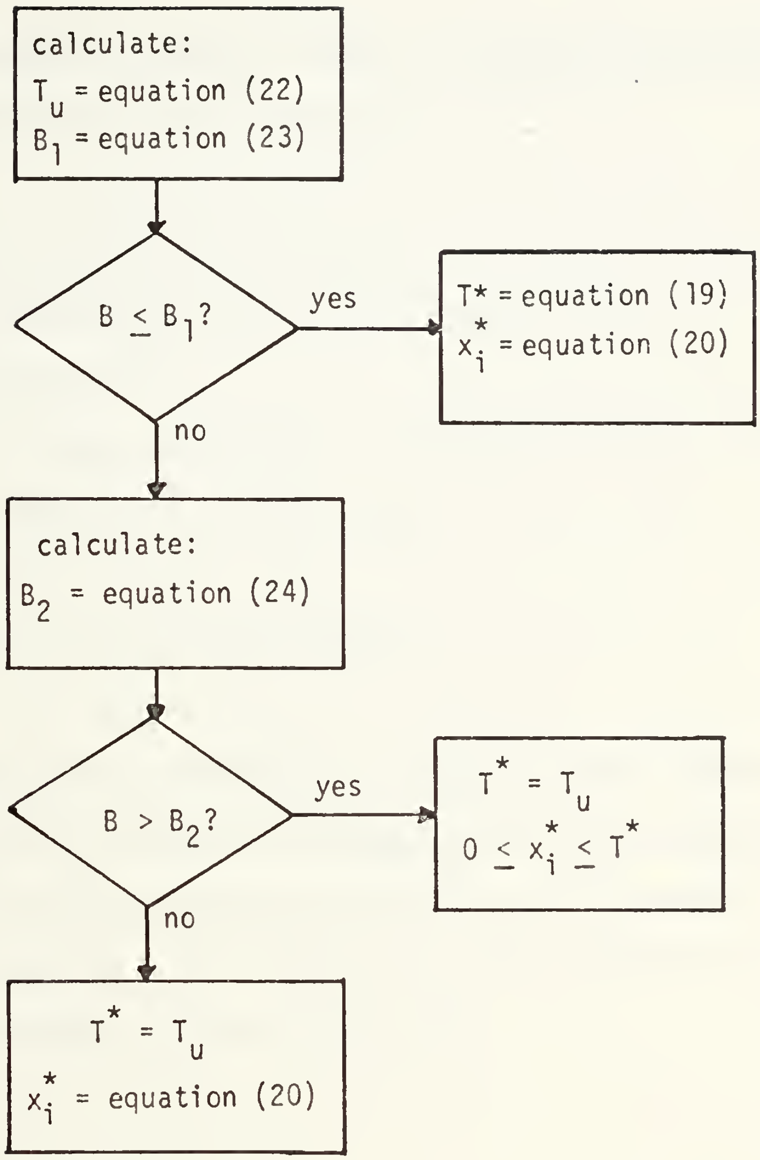


FIGURE 3: Flowchart for solving lot size model problems having $n > 2$ items.

Example 2

Suppose the data of example number 1 remains unchanged except for B. Assume B is now 600 ft^3.

The values for T_u and B_1 are unchanged. But since B is now greater than B_1, evaluation of B_2 is required. It is calculated to be 719 ft^3.

Because $B_1 < B < B_2$, $T^* = T_u = 0.41$ years and x_2 lies in the interval given by (17); that is,

$$0.12 \text{ years} \leq x_2 \leq 0.25 \text{ years} .$$

Checking the feasibility at time $t = 0$ shows that, using the lower bound as the x_2^* value, the occupied warehouse space would be 500 ft^3; using the upper bound as the x_2^* value, 600 ft^3 would be needed. The respective space needs for time $t = x_2^*$ would be 600 ft^3 and 467 ft^3.

D. COMMENTS

The constrained multi-item lot size system developed in this chapter differs from that of Holt [6] in the following ways:

1. Holt's model allows a different cycle length for each item, while a common cycle length for all products is used in the above model.
2. Holt's constraint required that the sum of average inventories be no greater than the space available, while

the constraint function used in the above model requires that the space occupied by the inventory at any time be no greater than the total space available.

3. The time phasing between orders for different items is calculated in the above model, while the time at which an order arrives is not regulated by Holt.

The second aspect is perhaps the most important difference because the actual space restriction in Holt's model is a nebulous notion. Realistically, the space constraint restricts the maximum on-hand inventory rather than the average level.

It is interesting to note that the solution for the scheduling period can also be used to calculate a minimum amount of storage space needed given a specific scheduling period. The equation for the warehouse space in terms of the scheduling period is

$$B = T D / \sum_{i=1}^{n} b_i r_i$$

where D is given by equation (21).

IV. A CONSTRAINED MULTI-ITEM ORDER LEVEL SYSTEM

A. THE MODEL

The system which is analyzed in this chapter is an n-item order level or periodic review system which is constrained by the total amount of space which is available for storage of the items. The analysis of this system will be conducted under the following assumptions:

1. Demand for item i is deterministic at a constant rate of r_i units per time.
2. The period between replenishments or scheduling period is a fixed constant T.
3. The replenishment quantity of the ith item raises the inventory at the time the quantity is received to the order level S_i after backorders are made up.
4. The replenishment rate is infinite.
5. The lead time is constant.
6. The same type of storage is used by all items.
7. The inventory carrying cost is proportional to the average amount in inventory. Its dimension is $ per unit time.
8. The backorder cost is proportional to the average shortage level. Its dimension is $ per unit time.
9. The order costs are ignored because they are a fixed cost if the scheduling period is a prescribed constant.

From these assumptions it can be seen that the lot sizes which are being considered are constant since

$$q_i = r_i T ,$$

where q_i represents the ith item lot size.

The variables which are to be optimized in this system are the order levels (S_i for the ith item) and the points in time at which each item arrives (x_i for the ith item, where $x_1 = 0$).

The cost for one scheduling period for the ith item as a function of the order level can be shown to be [7]:

$$C(S_i) = \frac{c_{1i} S_i^{\,2}}{2r_i T} + \frac{c_{2i}(r_i - S_i)^2}{2r_i T} ,$$

where c_{1i} = inventory carrying cost,

and c_{2i} = inventory shortage cost (using Naddor's notation).

Then the total cost equation for an n-item model can be expressed as:

$$C(S_1, S_2, \ldots, S_n) = \sum_{i=1}^{n} \left(\frac{c_{1i} S_i^{\,2}}{2r_i T} + \frac{c_{2i}(r_i - S_i)^2}{2r_i T} \right)$$

The storage constraint can be formulated as:

$$\sum_{i=1}^{n} b_i I_i(t) \leq B \quad \text{for all } t = 0 ,$$

where B = total space available,

b_i = amount of space which is occupied by one unit of item i,

$I_i(t)$ = inventory level of the ith item at any point in time t.

As in the preceding model, the maximum amount in inventory of any item occurs immediately after the ordered item arrives. Therefore the arrival time is a critical time with regard to the space constraint. Because a constraint will be required for the time of arrival of each item, there will be n constraints for n items.

Figure 4 illustrates the two possibilities in fluctuations of inventory for two items. Two cases are possible because, at the end of each scheduling period, an order for one item arrives and the other item is either on backorder or in stock. Care must be taken in formulating the constraints because when an item is on backorder its inventory level is negative but the amount of its occupied space is zero.

B. SOLUTIONS FOR THE TWO-ITEM CASE

The problem for the two-item case can be stated as follows:

Minimize

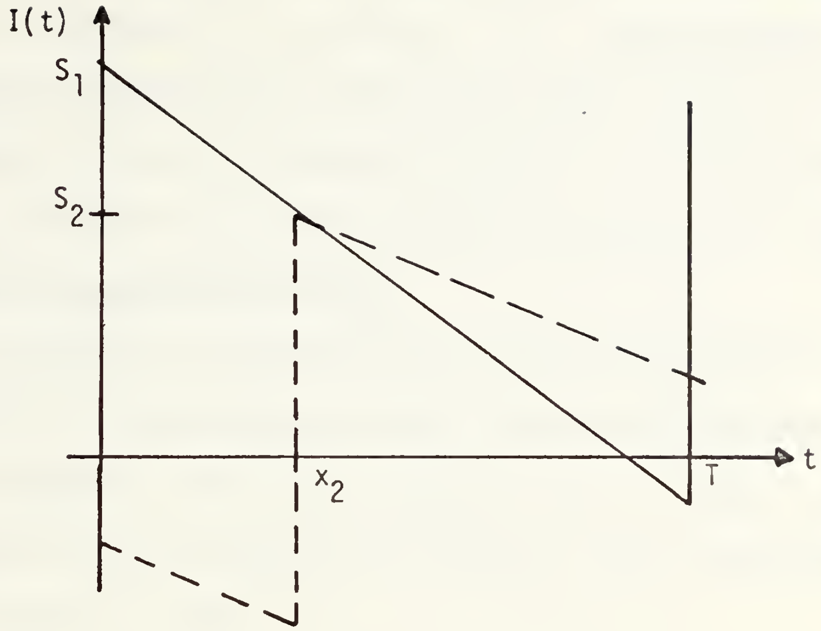
$$TC = \frac{c_{11}S_1^{\,2}}{2r_1T} + \frac{c_{21}(r_1T - S_1)^2}{2r_1T} + \frac{c_{12}S_2^{\,2}}{2r_2T} + \frac{c_{22}(r_2T - S_2)^2}{2r_2T} \quad ; \tag{25}$$

subject to the constraints

$$b_1S_1 + \max[0,\ b_2(S_2 - r_2(T-x_2))] \leq B \quad ; \tag{26}$$

$$b_2S_2 + \max[0,\ b_1(S_1 - r_1x_2)] \leq B \quad . \tag{27}$$

Case 1



Case 2

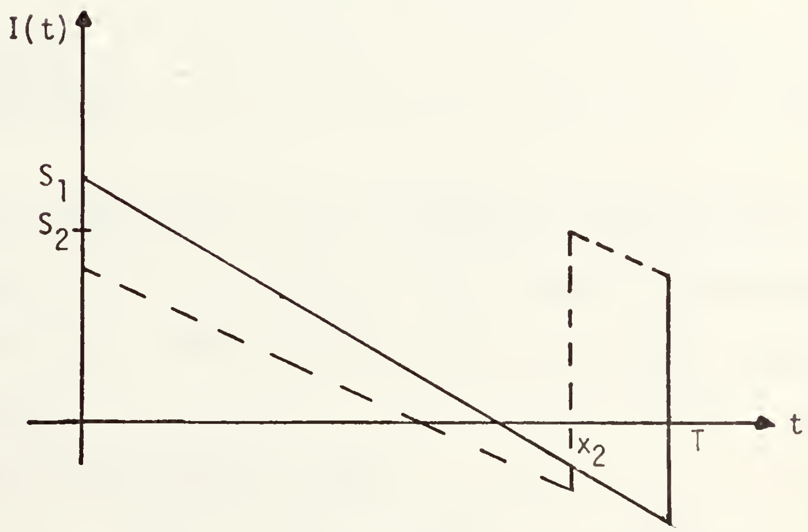


FIGURE 4: Fluctuation of inventory levels for two items for the deterministic order level model.

Constraint (26) reflects the combination of the constraint cases at t = 0. Whenever $S_2 - r_2(T-x_2) < 0$, which means that item number 2 is on backorder, this constraint reduces to the restriction that b_1S_1 be less than or equal to B. Similarly, constraint (27) reflects the combination of cases at $t = x_2$.

1. <u>Totally Unconstrained Case</u>

If the problem is unconstrained, then the optimal value for S_1 and S_2 can be obtained by simply taking the partial derivatives of equation (25) with respect to S_1 and S_2, setting them equal to zero and solving for S_1 and S_2. The solutions, denoted as S_{u1} and S_{u2}, are:

$$S_{u1} = \frac{c_{21}r_1T}{c_{11}+c_{21}} \quad ; \tag{28}$$

$$S_{u2} = \frac{c_{22}r_2T}{c_{12}+c_{22}} \quad ; \tag{29}$$

which are the same expressions as in Naddor's unconstrained case [7]. Because $B \geq b_1S_{u1} + b_2S_{u2} \equiv B_3$, even simultaneous arrivals are not space constrained and x_2 can assume any value in the interval $0 \leq x_2 \leq T$.

2. <u>Partially Unconstrained Case</u>

When $B < B_3$, then simultaneous arrivals would violate the space constraint. If B is not too restrictive, the problem could still be unrestricted relative to S_{u1} and/or S_{u2} if careful time phasing of the orders is done.

The individual constraints were:

$$b_1S_1 + \max[0,\ b_2(S_2 - r_2(T-x_2))] \leq B \quad ;$$

$$b_2S_2 + \max[0,\ b_1(S_1 - r_1x_2)] \leq B \quad .$$

Notice that S_{u1} and S_{u2} will not necessarily violate the space constraint if $B \geq B_2$ and x_2 is chosen carefully, where

$$B_2 = \max[b_1S_{u1},\ b_2S_{u2}] \quad . \tag{30}$$

They will therefore remain optimal.

If we assume that both $b_1S_{u1} < B$ and $b_2S_{u2} < B$ then the functions $\max[0,\ b_2(S_{u2} - r_2(T-x_2))]$ and $\max[0,\ b_1(S_{u1} - r_1x_2)]$ at $t = 0$ and $t = x_2$, respectively, could be positive. At $t = 0$ the value of the max function is not to exceed the total space available minus the space occupied by item number 1. A feasible positive max function at $t = 0$ satisfies:

$$0 < b_2(S_{u2} - r_2(T-x_2)) \leq B - b_1S_{u1} \quad ,$$

or, in terms of x_2,

$$T - \frac{S_{u2}}{r_2} < x_2 \leq T - \frac{S_{u2}}{r_2} + \frac{B - b_1S_{u1}}{b_2r_2} \tag{31}$$

If $x_2 \leq T - \frac{S_{u2}}{r_2}$ then $b_2(S_{u2} - r(T-x_2)) \leq 0$. If x_2 is greater than the right-hand bound then the total inventory at $t = 0$ would exceed B, the storage restriction value.

When $t = x_2$ the value of the positive max function is not to exceed the total space available minus the space occupied by the optimum order level of item number 2. This means that

$$0 < b_1(S_{u1} - r_1x_2) \leq B - b_2S_{u2} \quad ,$$

or, in terms of x_2,

$$\frac{S_{u1}}{r_1} - \frac{B - b_2S_{u2}}{r_1b_1} \leq x_2 < \frac{S_{u1}}{r_1} \; . \qquad (32)$$

In this case if $x_2 \geq \frac{S_{u1}}{r_1}$, then $b_1(S_{u1} - r_1x_2) \leq 0$ and if x_2 < left-hand bound, the total on-hand inventory at $t = x_2$ would exceed the storage restriction.

In comparing inequalities (31) and (32) it is clear that the space constraint will be only met if:

$$\frac{S_{u1}}{r_1} - \frac{B - b_2S_{u2}}{r_1b_1} \leq x_2 \leq T - \frac{S_{u2}}{r_2} + \frac{B - b_1S_{u1}}{b_2r_2} \quad . \qquad (33)$$

However, feasibility requires that the left-hand side of (33) be less than or equal to the right-hand side.

The two bounds of (33) are equal when $B = B_x$ where

$$B_x = B_3 - B_0 \quad , \tag{34}$$

and

$$B_3 = b_1 S_{u1} + b_2 S_{u2} \ ; \tag{35}$$

$$B_0 = [\ \frac{b_1 r_1 b_2 r_2}{b_1 r_1 + b_2 r_2}\ ]\ T \quad . \tag{36}$$

Therefore, if $B_2 \le B_x \le B < B_3$ then the problem is unconstrained with respect to S_1 and S_2 provided x_2^* is chosen within the boundaries of inequality (33).

Simplifying (33) results in:

$$\frac{B_3 - B}{b_1 r_1} \le x_2 \le T - \frac{B_3 - B}{b_2 r_2} \quad . \tag{37}$$

It follows that when $B = B_3$, the range given by (37) is $0 \le x_2 \le T$; when $B < B_3$ the range becomes narrower with the upper bound decreasing at a rate $1/b_2 r_2$ and the lower bound increasing at a rate $-1/b_1 r_1$ per one unit decrease in the value of B. We note in passing that if $B = b_1 S_{u1} = b_2 S_{u2}$, the range (33) reduces to

$$\frac{S_{u1}}{r_1} \le x_2 \le T - \frac{S_{u2}}{r_2} \ .$$

The condition resulting in $B_x > \max[b_1S_{u1}, b_2S_{u2}]$ is a case where both constraints are binding for any $B < B_x$ while the max functions are both still positive for B near B_x.

3. Constrained Cases

a. $B_x > B_2$ and $B_0 < B < B_x$.

If $B_x > \max[b_1S_{u1}, b_2S_{u2}]$ and $B < B_x$ the constraints when the max functions are both positive are:

$$b_1S_1 + b_2S_2 - b_2r_2T + b_2r_2x_2 \leq B \quad ;$$

$$b_2S_2 + b_1S_1 - b_1r_1x_2 \qquad \leq B \quad .$$

Subtracting the second inequality from the first gives:

$$x_2^* = \left[\frac{b_2r_2}{b_1r_1 + b_2r_2} \right] T = \frac{B_0}{b_1r_1} \quad . \tag{38}$$

Thus, x_2^* is independent of B provided that both max functions remain positive. Substitution of x_2^* into either constraint gives:

$$b_1S_1 + b_2S_2 \leq B + b_1r_1x_2^* \quad ,$$

$$\leq B + B_0 \quad .$$

Therefore we have the very simple relationship that, whatever values of S_1 or S_2 are chosen, we must have

$$b_1S_1^* + b_2S_2^* \leq B + B_0 \quad . \tag{39}$$

Now inequality (39) does not reflect the fact that the space constraints at $t = 0$ and $t = x_2$ would be violated if either $b_1S_1^* > B$ or $b_2S_2^* > B$. Therefore, the following conditions must also be imposed:

$$b_1S_1^* \leq B \quad ; \tag{40}$$

$$b_2S_2^* \leq B \quad . \tag{41}$$

The form of the resulting constraint region is illustrated in Figure 5(a). The unconstrained optimal solution (S_{u1}, S_{u2}) lies outside this region. Because the objective function is convex in S_1 and S_2 and the constraint region is convex, it follows that the constrained optimum lies on the boundary of the feasible region.

If B is reduced to B_0 we get the condition illustrated in Figure 5(b) and when $B < B_0$ we get the result of Figure 5(c). These cases will be analyzed in the next section.

To search for the optimal solution when $B_0 < B < B_x$ we evaluate S_1 and S_2 using only one segment of the boundary at a time. We begin by assuming that the constrained S_1^* and S_2^* lie on the line given by (39) with equality. We can then use (39) to solve for S_2 as a function of S_1, substitute that function into the objective function, differentiate with respect to S_1, set the derivative to zero and solve for S_1. We get

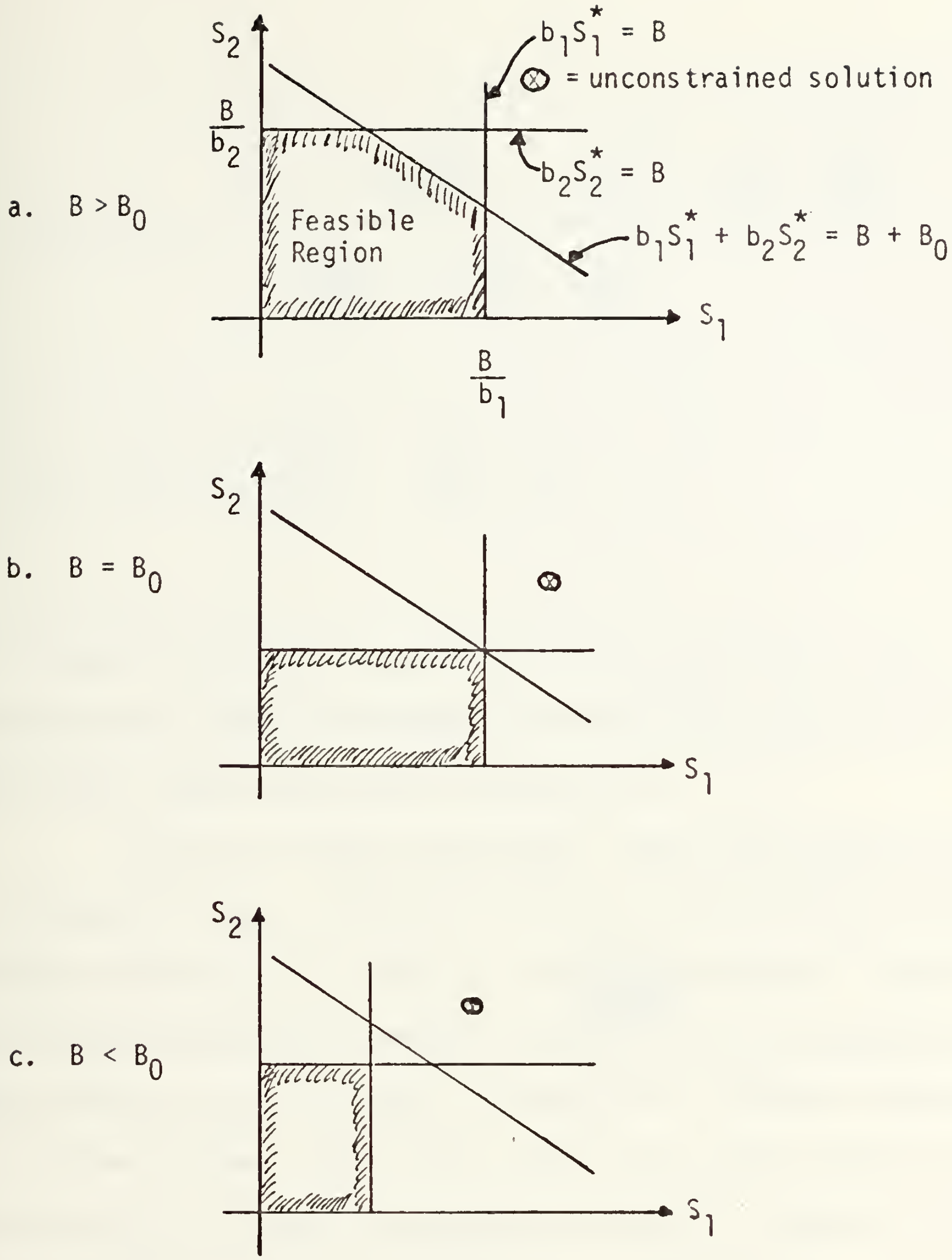


FIGURE 5: Feasible regions for the two-item deterministic order level model where $b_1S_1^* + b_2S_2^* \leq B + B_0$.

$$S_1^* = \hat{S}_1 \equiv \frac{c_{21} - c_{22}\,[\frac{b_1}{b_2}] + [\frac{B + B_0}{b_2}]\,[\frac{b_1}{b_2}]\,[\frac{c_{22}}{S_{u2}}]}{[\frac{c_{21}}{S_{u1}}] + [\frac{b_1}{b_2}]^2\,[\frac{c_{22}}{S_{u2}}]} \; ; \tag{42}$$

$$S_2^* = \hat{S}_2 \equiv [\frac{B + B_0}{b_2}] - [\frac{b_1}{b_2}]\,\hat{S}_1 \quad . \tag{43}$$

If either $\hat{S}_1$ or $\hat{S}_2$ violates the $b_1 S_1 \leq B$ or $b_2 S_2 \leq B$ constraints then the optimal constrained values lie on either of these latter constraint boundaries.

If $\hat{S}_1$ is larger than B/b_1 then the largest feasible S_1 value is B/b_1. This then will be the optimal value of S_1. Substitution of $S_1^* = B/b_1$ into the objective function and differentiating with respect to S_2 will give S_{u2} when the derivative is set to zero. Therefore, optimal constrained S_2 is the largest value the constraints will allow without reducing S_1^*. This corresponds to the S_2 value at the intersection of (39) and (40); that is $S_2^* = B_0/b_2$.

Similar arguments lead to $S_1^* = B_0/b_1$ and $S_2^* = B/b_2$ when $\hat{S}_2$ is greater than B/b_2.

In summary, if $B_0 < B < B_x$ then x_2^* is given by equation (38). To determine S_1^* and S_2^*:

1. Calculate $\hat{S}_1$, $\hat{S}_2$.
2. If $\hat{S}_1 \leq B/b_1$, $\hat{S}_2 \leq B/b_2$ then $S_1^* = \hat{S}_1$, $S_2^* = \hat{S}_2$.

3. If $\hat{S}_1 > B/b_1$, then $S_1^* = B/b_1$, $S_2^* = B_0/b_1$.

4. If $\hat{S}_2 > B/b_2$, then $S_1^* = B_0/b_1$, $S_2^* = B/b_2$.

b. $B_x > B_2$ and $B \leq B_0$.

Figure 5(b) and 5(c) emphasize that when $B \leq B_0$ the constraint given by (35) lies outside the boundaries of the feasible region and is trivially satisfied by any solution in the feasible region. It follows from the arguments in the preceding section (a) that $S_1^* = B/b_1$ and $S_2^* = B/b_2$ and we realize immediately that both max functions must be zero.

The analysis of the preceding section showed that if

$$\frac{S_1}{r_1} \leq x_2 \leq T - \frac{S_2}{r_2} ,$$

then both max functions will remain zero. Substitution of S_1^* and S_2^* into this range gives

$$\frac{B}{b_1 r_1} \leq x_2 \leq T - \frac{B}{b_2 r_2} . \tag{44}$$

This range reduces to $x_2 = B_0/b_1 r_1$ (equation (38)) when $B = B_0$; the range increases to $0 \leq x_2 \leq T$ as B is reduced to zero.

Figure 6 summarizes the ranges of optimal x_2 for the case where $B_x > B_2$. From this figure it becomes obvious that $x_2 = B_0/b_1 r_1$ is optimal regardless of the value of B.

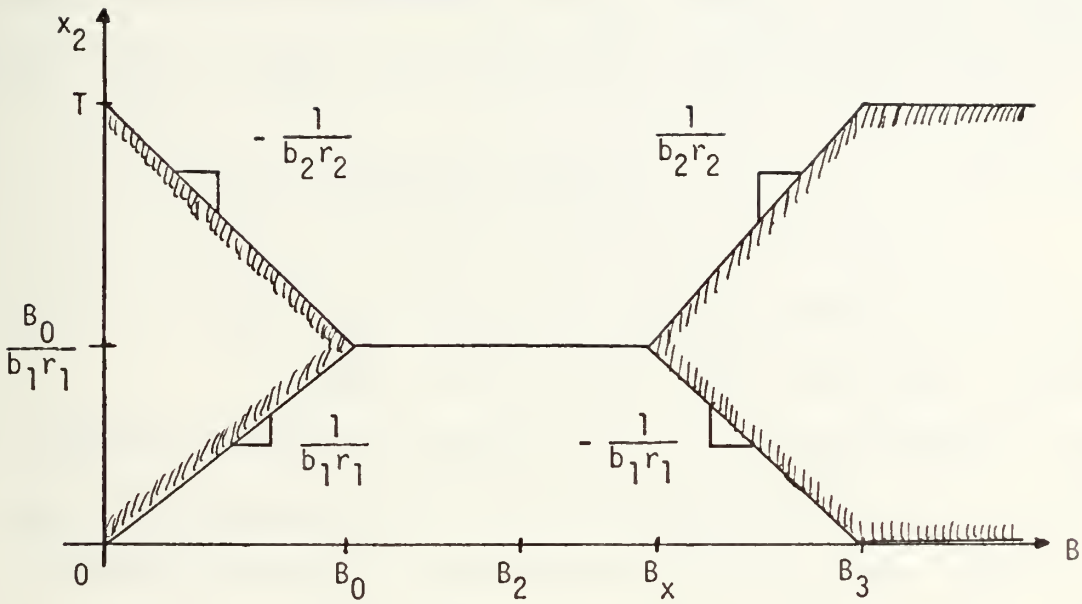


FIGURE 6: The range of optimal x_2 as a function of B for the case where $B_x > B_2$.

c. $B_2 > B_x$ and $B < B_2$.

In the constrained case where $B_2 > B_x$, further reduction into cases where $b_1S_{u1} > b_2S_{u2}$, $b_2S_{u2} > b_1S_{u1}$ and $b_1S_{u1} = b_2S_{u2}$ are necessary. We will also make use of

$$B_1 \equiv \min[b_1S_{u1}, b_2S_{u2}] \quad . \tag{45}$$

(1) $B_2 = b_1S_{u1}$, $B_1 = b_2S_{u2}$, and $B_1 < B < B_2$.

When $B_1 < B < B_2$ then the first constraint to become binding is that at t = 0. Its max function will be zero. An appreciation of what is taking place can be obtained by looking at the x_2 range given by (33). When $B_2 > B_x$ then the upper and lower bounds do not reduce to the same value (at $B = B_x$) before the $B - b_1S_{u1}$ term in the upper bound goes to zero. The resulting range for x_2 from (33) is now:

$$\frac{b_2S_{u2}}{b_1r_1} \leq x_2 \leq T - \frac{S_{u2}}{r_2} \tag{46}$$

when $S_1^* = B/b_1$ is substituted for S_{u1}. The lower bound is only valid however as long as $B \geq B_1 \equiv b_2S_{u2}$. The optimal value of S_2 is S_{u2}.

Inspection of (46) shows that it is independent of B provided that $B_1 < B < B_2$. When the value of B is reduced to B_1, (46) reduces to (44); that is

$$\frac{B}{b_1r_1} \leq x_2 \leq T - \frac{B}{b_2r_2} \quad ,$$

and, since S_2^* must be B/b_2 for all $B < B_1$, this range must be valid for all such B.

The analysis in section b above showed that (44) gave feasible x_2 values as long as $B \leq B_0$. The analysis in this section then suggests that B_1 must be less than or equal to B_0 if (46) and (44) are to be feasible and identical at $B = B_1$. Now the range of (46) is feasible if

$$\frac{b_2 S_{u2}}{b_1 r_1} \leq T - \frac{S_{u2}}{r_2} \quad .$$

Collecting like terms results in

$$b_2 S_{u2} \leq \left[\frac{b_1 r_1}{b_1 r_1 + b_2 r_2} \right] T \quad ,$$

which reduces to $B_1 \leq B_0$.

In summary, if $B_2 \equiv b_1 S_{u1} > B_x$ and $B_1 \equiv b_2 S_{u2} < B < B_2$ then x_2^* is any value in the range given by (46) and $S_1^* = B/b_1$, $S_2^* = S_{u2}$. When $B \leq B_1$ we again have x_2^* given by (44) and $S_1^* = B/b_1$, $S_2^* = B/b_2$.

Figure 7 shows the range of optimal x_2 as a function of B for this case. It is immediately evident that $x_2 = B_0/b_1 r_1$ is optimal regardless of the value of B.

(2) $B_2 = b_2 S_{u2}$, $B_1 = b_1 S_{u1}$ and $B_1 < B < B_2$.

By similar arguments to those given above in section (1) we get $S_1^* = S_{u1}$ and $S_2^* = B/b_2$. The x_2^* range from (33) reduces to

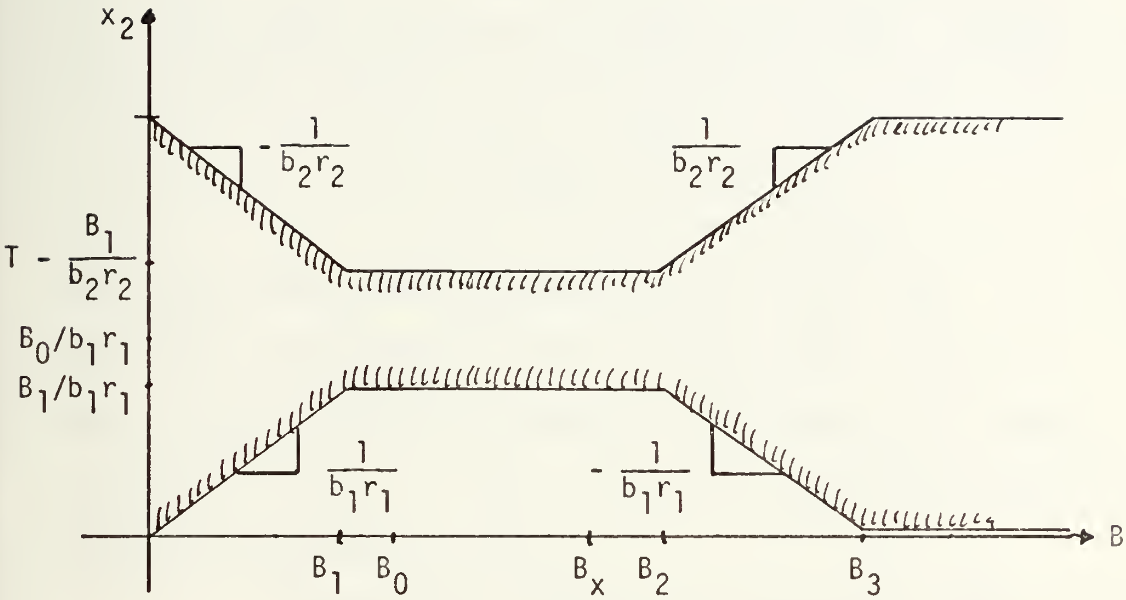


FIGURE 7: The range of optimal x_2 as a function of B for the case where $B_x < B_2$ and $B_1 < B_2$.

$$\frac{S_{u1}}{r_1} \leq x_2 \leq T - \frac{b_1 S_{u1}}{b_2 r_2} \quad , \tag{47}$$

when $B_1 < B < B_2$. When $B \leq B_1$, $S_1^* = B/b_1$, $S_2^* = B/b_2$ and x_2^* can be obtained from (44). Figure 7 also illustrates the range of optimal x_2 values for this case.

(3) $B_1 = b_1 S_{u1} = b_2 S_{u2} = B_2$

In this case, the interval $B_1 < B < B_2$ does not exist and therefore we have $S_1^* = S_{u1}$, $S_2^* = S_{u2}$ and the x_2 interval is given by (37) when $B_1 = B_2 < B < B_3$. When $B \leq B_1$ then $S_1^* = B/b_1$, $S_2^* = B/b_2$ and the x_2 interval is given by (44). Figure 8 shows the optimal range of x_2 as a function of B.

Having now examined all possible cases it can be seen that there are three major possibilities:

First: the problem is unconstrained if $B \geq B_3$ and partially unconstrained (the range of x_2 is limited) if max $[B_2, B_x] < B < B_3$. Solutions for S_1^* and S_2^* are the economic order levels S_{u1} and S_{u2}.

Second: If the storage constraint is between $\min[B_1, B_0]$ and $\max[B_2, B_x]$ then the solutions depend on the relationship of the parameters to each other and to B. They can be classified according to two ranges for B: namely, $B_0 < B < B_x$ and $B_1 < B < B_2$.

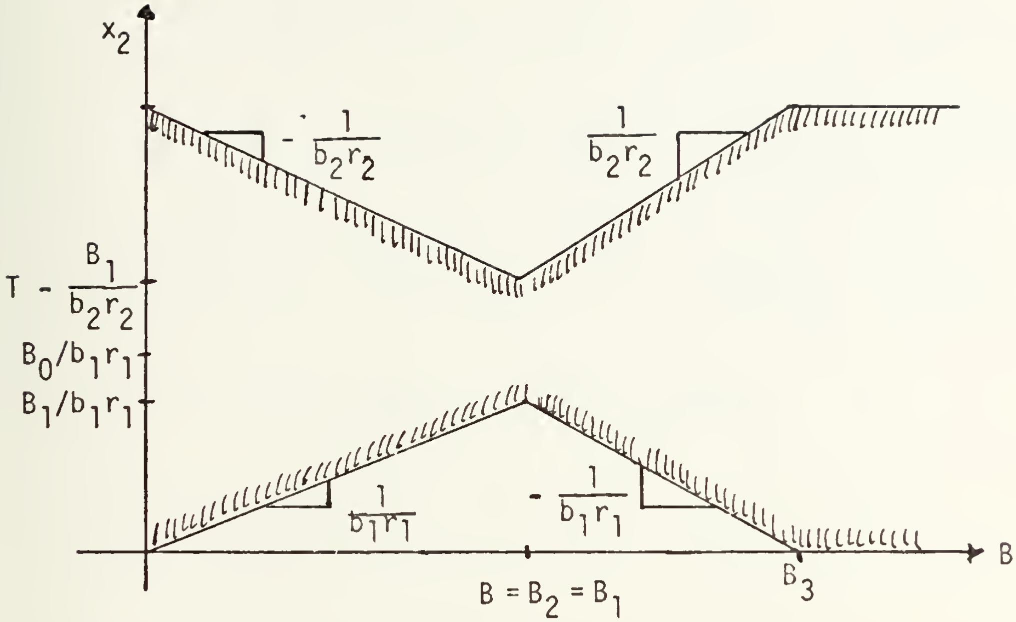


FIGURE 8: The range of optimal x_2 as a function of B for the case where $B_x < B_2$ and $B_1 = B_2$.

Solutions for S_1^* and S_2^* are:

1. $S_1^* = \hat{S}_1 \quad ; \quad S_2^* = \hat{S}_2$

2. $S_1^* = B/b_1 \quad ; \quad S_2^* = B_0/b_2$

3. $S_1^* = B_0/b_1 \quad ; \quad S_2^* = B/b_2$

4. $S_1^* = S_{u1} \quad ; \quad S_2^* = B/b_2$

5. $S_1^* = B/b_1 \quad ; \quad S_2^* = S_{u2}$

The first three solutions correspond to the $B_0 < B < B_x$ case and x_2^* can only have one value. The last two solutions correspond to the $B_1 < B < B_2$ case and x_2^* can take on any values within a computed interval.

Third: In all cases where $B \leq \min[B_1, B_0]$ the space constraint is so restrictive that only one item can be stored at a time. Hence $S_1^* = B/b_1$ and $S_2^* = B/b_2$, and the interval for x_2 expands as B decreases.

The details of the optimal solutions of the two-item case are summarized by the flowchart presented in Figure 9.

C. THE n-ITEM CASE

The n-item case can be easily formulated using the same approach as for the two-item case. The general n constraint equations can be written for the times at which the items arrive using the max functions.

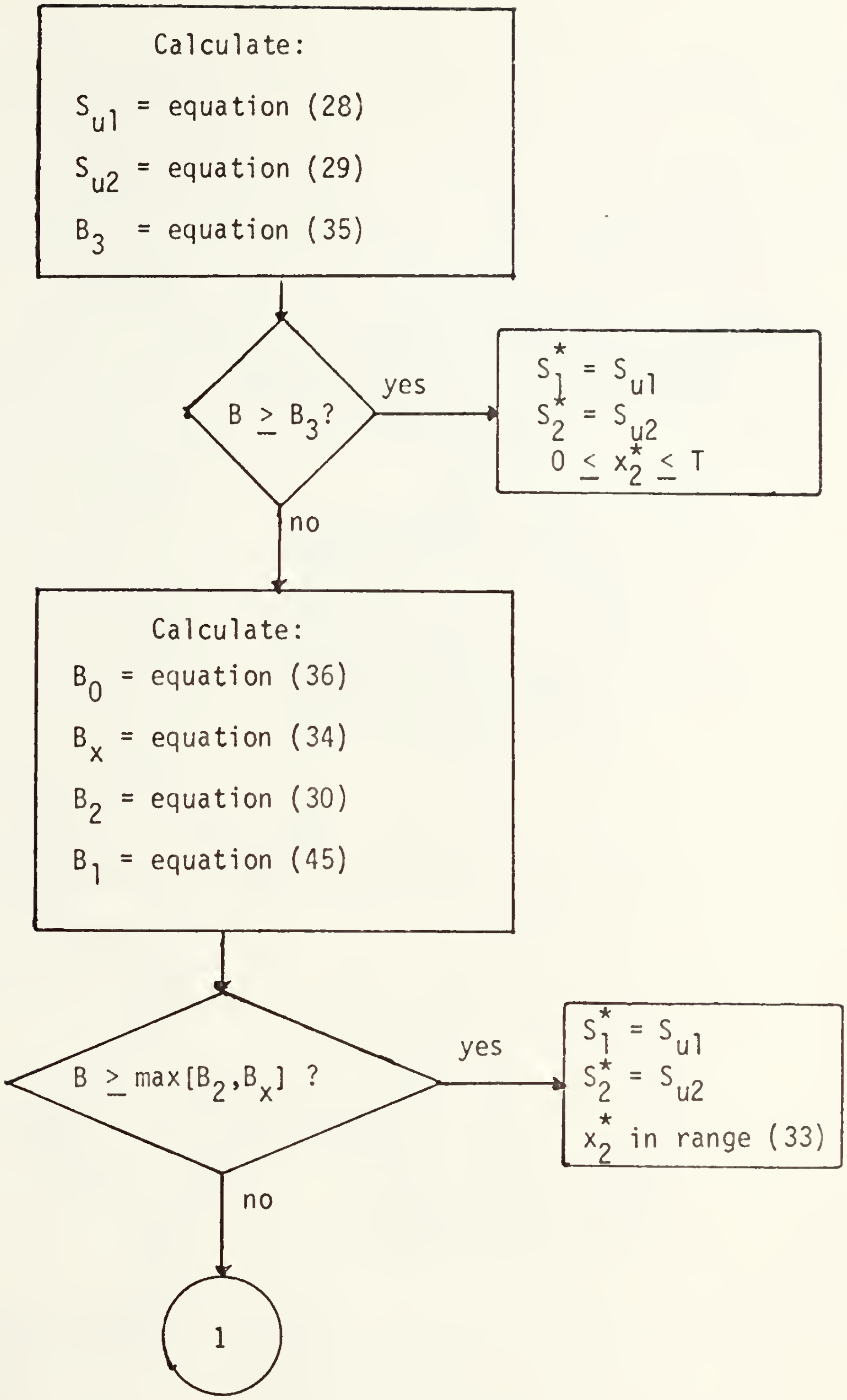


FIGURE 9: Flowchart for determining S_1^*, S_2^* and x_2^* for the two-item deterministic order level model.

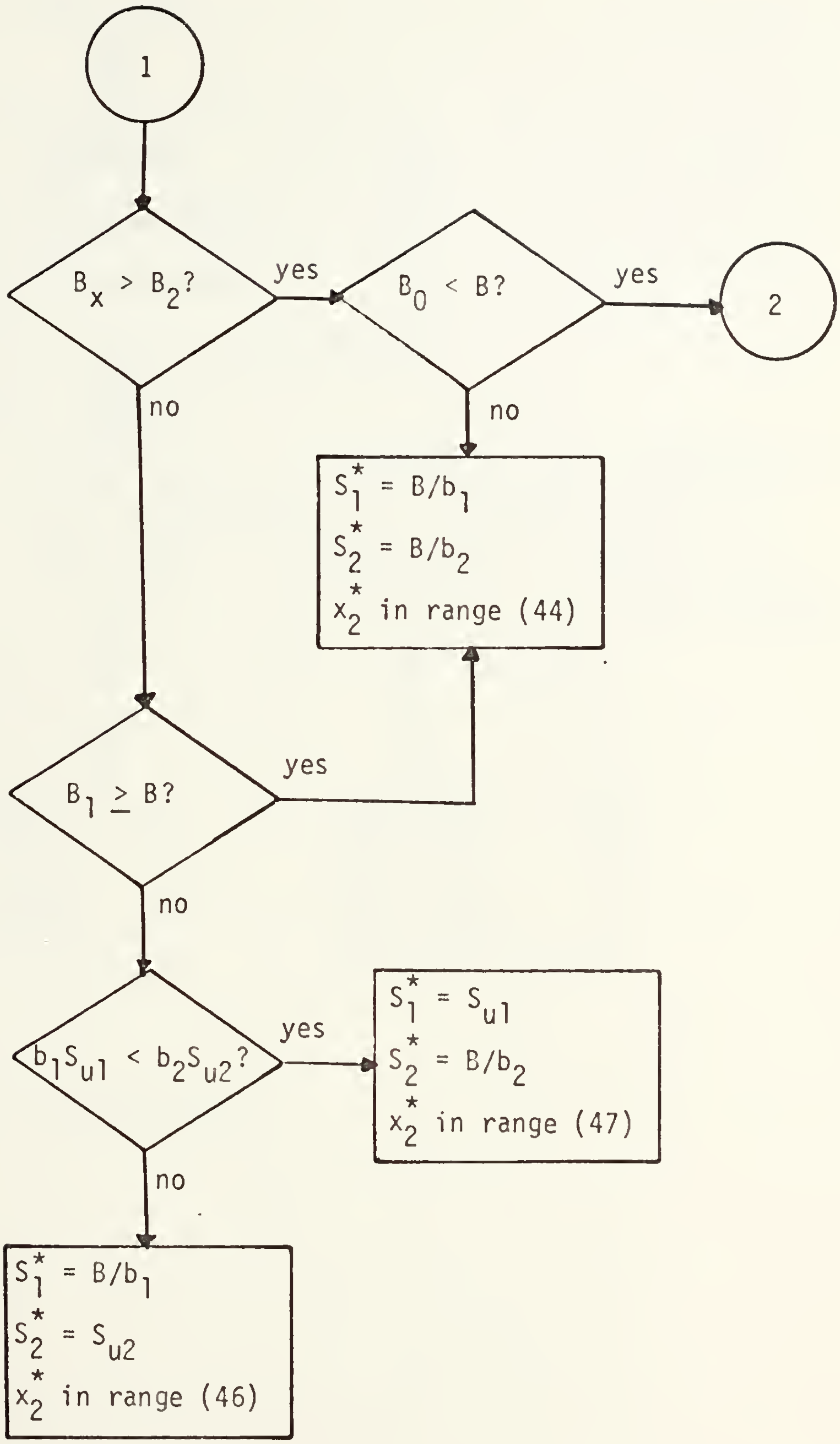


FIGURE 9: (continuation)

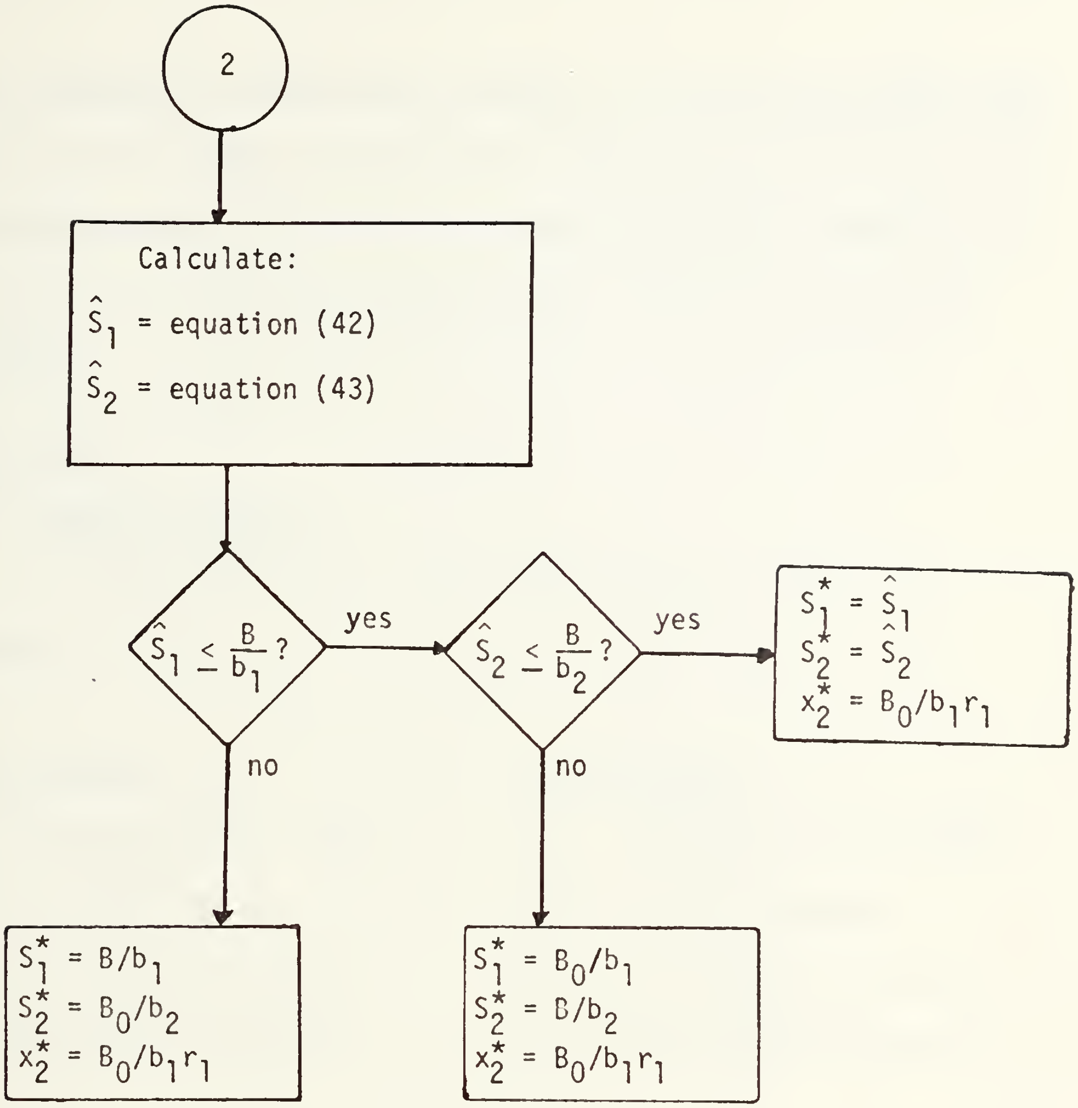


FIGURE 9: (continuation)

The principle difficulty involved in trying to solve this problem is the number of possible combinations which must be considered because of the various max functions. Because of its complexities the n-item case will not be addressed in this thesis.

D. EXAMPLES

1. Example Number 1

A warehouse stores two items with the following parameters:

Item Number 1:	Item Number 2:
$r_1 = 200$ units per year;	$r_2 = 250$ units per year;
$b_1 = 5\ ft^3$/unit;	$b_2 = 3\ ft^3$/unit;
$c_{11} = \$\ 2$ per unit-year;	$c_{12} = \$\ 1$ per unit-year;
$c_{21} = \$30$ per unit-year;	$c_{22} = \$25$ per unit-year.

Total warehouse space available is 600 ft^3; the scheduling period T is one year. How many units of each item should be ordered and when should the orders arrive?

Following the steps of Figure 9, we first calculate S_{u1}, S_{u2} and B_3.

$$S_{u1} = 187.5 \quad ; \quad S_{u2} = 240.4 \quad ; \quad B_3 = 1658 \ .$$

Since $B = 600 < B_3$, we must calculate B_0, B_x, B_1 and B_2.

$$B_0 = 429 \;;\quad B_x = 1229 \;;\quad B_1 = 721 \;;\quad B_2 = 937 \;.$$

We see that $B_x > B_2$ and $B_0 < B$ and we will have to calculate $\hat{S}_1$ and $\hat{S}_2$.

$$\hat{S}_1 = 106.4 \quad;\quad \hat{S}_2 = 165.6 \;.$$

Now, because $\hat{S}_1 < \frac{B}{b_1} = 120$ and $\hat{S}_2 < \frac{B}{b_2} = 200$, the optimal values for S_1 and S_2 are $\hat{S}_1$ and $\hat{S}_2$. The corresponding x_2 value is given by equation (38). In summary, the optimal solutions are:

$$S_1^* = 106.4 \text{ units} \quad;\quad S_2^* = 165.6 \text{ units} \;;$$

$$x_2^* = 0.429 \text{ years} \quad .$$

Total costs TC, using equation (25), are \$ 1125 per year.

2. Example Number 2

The available warehouse space is the same as in Example number 1. However, the parameters of the items are:

Item Number 1:	Item Number 2:
$r_1 = 200$ units per year;	$r_2 = 250$ units per year;
$b_1 = 5\ ft^3/unit$;	$b_2 = 3\ ft^3/unit$;
$c_{11} = \$\ 5$ per unit-year;	$c_{12} = \$\ 1$ per unit-year;
$c_{21} = \$100$ per unit-year;	$c_{22} = \$\ 3$ per unit-year.

Calculating S_{u1}, S_{u2}, and B_3:

$$S_{u1} = 190 \ ; \quad S_{u2} = 187.5 \ ; \quad B_3 = 1512 \ .$$

Since $B = 600$ is less than B_3 calculations of B_0, B_x, B_1, and B_2 are required.

$$B_0 = 429 \ ; \quad B_x = 1083 \ ; \quad B_1 = 562.5 \ ; \quad B_2 = 950 \ .$$

Because $B_x > B_2$ and $B_0 < B$ we have to calculate $\hat{S}_1$ and $\hat{S}_2$.

$$\hat{S}_1 = 182.5 \quad ; \quad \hat{S}_2 = 38.8 \ .$$

Because $\hat{S}_1 > \frac{B}{b_1} = 120$, $\hat{S}_1$ is not a feasible value for S_1. The optimal feasible solutions are:

$$S_1^* = B/b_1 = 120 \text{ units} \ ; \quad S_2^* = B_0/b_2 = 143 \text{ units} \ ;$$

$$x_2^* = B_0/b_1 r_1 = 0.429 \text{ years.}$$

Total costs, using equation (25), are \$ 1890 per year.

E. COMMENTS

The major difference between the model examined by Naddor and the model just developed is that the different items in Naddor's model all arrive at the same time, while the model being considered in this paper permits time phasing of the arrival time for each item. This time phasing allows a higher average inventory level for each item and, consequently, a reduced average shortage level and a lower total average annual cost for the system. For the data of Example number one the optimal order levels using Naddor's model are $S_1 = 51.3$ units per year and $S_2 = 114.5$ units per year, yielding total average annual costs of \$2600 (using Equation (25)). Time phasing therefore "saves" \$1475 per year.

In addition to Naddor's model being more expensive, the solutions are not always feasible. Naddor's analysis implies that a negative optimal order level "creates" additional available space. In the model of this paper this problem has been carefully considered in the max functions. Naddor does not address this problem at all! Using Naddor's results to solve Example number 2 we get $S_1 = 177$ units and $S_2 = -94$ units. Clearly, $b_1S_1 > B$ and hence $S_1 = 177$ units is infeasible.

V. SUMMARY AND RECOMMENDATIONS

A. SUMMARY

Chapter III provides a minimum cost (ordering plus holding) solution to a multi-item lot size inventory problem subject to a constraint on the maximum total space available for storing the inventory. The variables which were optimized were the scheduling period (time between orders of a single item) and the inter-order times between the different items. The assumptions which were made in the formulation of this model were a deterministic demand, infinite replenishment rate, no shortages allowed and a common scheduling period for all items.

Chapter IV provides a minimum cost (holding plus back-order) solution for a two-item system under periodic review and subject to the same space constraint as in Chapter III. The variables which were optimized were the order levels for the two items and the inter-order time between them. The solutions have several possible forms depending upon the interactions of the constraint and the parameters of the items. The assumptions which were made in the formulation of the problem were a deterministic demand, infinite replenishment rate and a prescribed interval between reviews which is identical for all items.

B. RECOMMENDATIONS

Basic to the models presented in this thesis is the assumption of deterministic demand. In reality, we are faced with random demand patterns and therefore it is impossible to determine with certainty the future states of an inventory system. However, it has been shown that examination of deterministic inventory models can provide a better understanding of the situation being modelled and may suggest approximations which would apply in the case of random demand [4]. This thesis provides such an understanding and one of the next steps towards improving existing severely constrained multi-item systems is to obtain demand data and classifications of items as either needing continuous reviewing or needing only a periodic review.

The analysis of the lot size model in Chapter III was conducted because there are a certain number of items which should never be out of stock in a BVP. Further research should be able to provide a listing of these items and a probability distribution to describe their underlying demand. Because these items should be continuously reviewed to avoid stockouts they should probably be stored together on one or several trucks to facilitate careful monitoring.

The periodic review model of Chapter IV was presented because the majority of the items which have to be stored by a BVP are not of such vital importance that they cannot be on backorder a certain amount of time. This model emphasized

the importance of knowing the values of time-dependent backorder costs. Such costs are needed and military essentiality coding should be a valuable help in appraising such costs for each item, thus insuring that the more important items have higher backorder costs and will therefore be less on backorder in an economic inventory system. The demand distribution will also be needed before the model results can be extended to a random demand situation.

Though the problems of solving the n-item case in Chapter IV seem to be severe for large n, situations involving large n values probably do not occur in a BVP. As has been already mentioned, a BVP consists of a fleet of trucks. If the items which are to be stored on one truck are selected in a way that all items have similar parameters — low cost and high demand for example — then it could be possible to form classes of items and treat each class as one item. Thus the classes per truck could perhaps be reduced to two or three. Grouping of items into classes of similar demand would also eliminate the need to determine a probability distribution for each item; the demand distribution for the class would suffice. Of course the notion of "similar demand" is rather nebulous and would require some further defining before grouping could be made.

While demand and cost data are being gathered, further analytical work should proceed using the ideas behind the approximate random demand models of Chapter 4 and 5 of Hadley

and Whitin [4]. In particular the (Q, r) model of Chapter 4 addresses the problem of optimal ordering when backorder costs are high. This situation corresponds to that of the BVP for items which should "never be out of stock".

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